S.-T. Yau College Student Mathematics Contests 2025

Computational and Applied Mathematics

(6 problems)

Problem 1 We consider the multipoint iteration method

$$x_{k+1} = x_k - \alpha \frac{f(x_k)}{f'\big(x_k - \beta f(x_k)/f'(x_k)\big)},$$

where α and β are arbitrary parameters, for solving the equation $f(x) = 0$. Determine the values α and β such that the multipoint method achieves the highest possible order of convergence for finding ξ, a simple root of $f(x) = 0$.

Problem 2 Compute the spectral radius of the matrix $\mathbf{A}^{-1}$, where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Problem 3 Let the ordinary Legendre polynomial of degree k be denoted $P_k(x)$ for $k \geq 0$. The associated Legendre functions are defined as

$$P_k^m = (-1)^m (1 - x^2)^{m/2} \frac{d^m}{dx^m} P_k(x), \quad m > 0, \quad k \geq m.$$

(Note that despite the name, for odd m they are not actually polynomials.)

(a) Consider the interpolation problem of finding coefficients a_k such that

$$\sum_{k=1}^{N} a_k P_k^1(x_i) = y_i, \quad i = 1, ..., N.$$

Prove that this linear system of equations for the unknown coefficients a_k is nonsingular provided that the interpolation points $\{x_i\}$ exclude ± 1 and are distinct.

(b) Consider the approximation problem of finding coefficients a_k to minimize the squared approximation error

$$\Big|\Big| f(x) - \sum_{k=1}^{N} a_k P_k^1(x) \Big|\Big|_2^2,$$

where the L^2 norm is taken over $x \in [-1, 1]$. Derive the linear system for the coefficients a_k and explain why it is nonsingular.

(c) Let $\mathbf{M}$ be the coefficient matrix from part (b). Prove that $\mathbf{M}_{k,j} = 0$ when $k + j$ is odd.

Problem 4 Given a set of column vectors $y_1, \ldots, y_n \in \mathbb{R}^m$, let $\mathcal{V} = \text{span}\{y_1, \ldots, y_n\} \subset \mathbb{R}^m$. How can we find $\ell \leq \dim \mathcal{V}$ orthonormal vectors $\{\psi_i\}_{i=1}^{\ell}$ in $\mathbb{R}^m$ that minimize

$$J(\psi_1, \ldots, \psi_\ell) = \sum_{j=1}^{n} \left\| y_j - \sum_{i=1}^{\ell} (y_j^\top \psi_i)\psi_i \right\|^2,$$

where $\|y\| = \sqrt{y^\top y}$ is the Euclidean norm?

Problem 5 Consider the initial-value problem

$$\frac{\partial u}{\partial t} + u = \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad 0 < t \leq T,$$
$$u(x, 0) = u_0(x), \quad -\infty < x < \infty$$

where T is a fixed positive real number, and u_0 is a real-valued continuous function on $\mathbb{R}$. Consider the θ-scheme

$$\frac{U_j^{m+1} - U_j^m}{\Delta t} + [\theta U_j^{m+1} + (1-\theta)U_j^m] = \theta \frac{U_{j+1}^{m+1} - 2U_j^{m+1} + U_{j-1}^{m+1}}{(\Delta x)^2} + (1-\theta)\frac{U_{j+1}^m - 2U_j^m + U_{j-1}^m}{(\Delta x)^2}$$

for $j \in \mathbb{Z}$, $m = 0, ..., M-1$, where $\Delta x > 0$ and $\Delta t = T/M$, $M \geq 1$, and $U_j^0 = u_0(j\Delta t)$, $j \in \mathbb{Z}$.

(a) Define the ℓ_∞-norm as $||U^m||_{\ell_\infty} := \max_{j\in\mathbb{Z}} |U_j^m|$, and assume that $||U^0||_{\ell_\infty}$ is finite. Prove that for $\theta \in [0, 1]$,

$$||U^m||_{\ell_\infty} \leq \left(\frac{1-(1-\theta)\Delta t}{1+\theta\Delta t}\right)^m ||U^0||_{\ell_\infty},$$

holds for all $1 \leq m \leq M$, provided that $A(\theta)\Delta t \leq \frac{(\Delta x)^2}{2+(\Delta x)^2}$, where $A(\theta)$ is a constant, depending on the choice of θ, which you should determine.

(b) Define the ℓ_2-norm as $||U^m||_{\ell_2} := \left(\Delta x \sum_{j\in\mathbb{Z}} |U_j^m|^2\right)^{1/2}$ and suppose that $||U^m||_{\ell_2}$ is finite.

- For $\theta \in [\frac{1}{2}, 1]$, show that $\|U^m\|_{\ell_2} \leq \|U^0\|_{\ell_2}$ holds for any $\Delta t, \Delta x > 0$ and all $1 \leq m \leq M$.
- For $\theta \in [0, \frac{1}{2})$, prove that $\|U^m\|_{\ell_2} \leq \|U^0\|_{\ell_2}$ under the condition $B(\theta)\Delta t \leq \frac{2(\Delta x)^2}{4+(\Delta x)^2}$, where $B(\theta)$ is a constant, depending on the choice of θ, which you should determine.

Problem 6 Consider the stiff system of ordinary differential equations:

$$\frac{d\mathbf{y}}{dt} = \mathbf{f}(t, \mathbf{y}), \quad \mathbf{y}(0) = \mathbf{y}_0$$

where $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, $\mathbf{y}_0 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, and

$$\mathbf{f}(t, \mathbf{y}) = \begin{pmatrix} -1000y_1 + 999y_2 \\ -y_2 \end{pmatrix}$$

(a) Find the exact solution $\mathbf{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$.

(b) For the explicit Euler method:

$$\mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{f}(t_n, \mathbf{y}_n)$$

determine the absolute stability region and prove divergence when $h > 0.002$.

(c) For the implicit Euler method:

$$\mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{f}(t_{n+1}, \mathbf{y}_{n+1})$$

prove unconditional stability for any $h > 0$.

(d) For the trapezoidal rule:

$$\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{h}{2}[\mathbf{f}(t_n, \mathbf{y}_n) + \mathbf{f}(t_{n+1}, \mathbf{y}_{n+1})]$$

analyze its stability.